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ON A HYPERBOLIC SYSTEM OF CONSERVATION
LAWS WHICH IS NOT STRICTLY HYPERBOLIC.

Tai-Ping Liu and Ching-Hua Wang

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Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

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ABSTRACT

We study a system of quasilinear hyperbolic conservation laws which is
hyperbolic but not strictly hyperbolic. Such systems arise naturally in
continuum mechanics such as elastic, multiphase flows. We are interested
mainly in the large time behavior of the solution. Due to the nonlinearity of
the system and the entropy condition, solutions converge to very simple
elementary waves. Nonstrict hyperbolicity of the system may cause a stronger
nonlinear interactions between waves pertaining to different families; in
particular, such interactions may regularize linear waves in the solution.
The solutions are constructed using the random choice method.

AMS (MOS) Subject Classifications: 35L65, 35B40, 35L67

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asymptotic behavior

Work Unit Number 1 (Applied Analysis)

*Department of Mathematics, University of Maryland, College Park, MD 20742

**Courant Institute of Mathematical Sciences, New York University, 251 Mercer
Street, New York, NY 10012

SIGNIFICANCE AND EXPLANATION

We study a system of quasilinear hyperbolic conservation laws which is
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continuum mechanics such as elastic, multiphase flows. We are interested
mainly in the large time behavior of the solution. Due to the nonlinearity of
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ON A HYPERBOLIC SYSTEM OF CONSERVATION
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Tai-Ping Liu* and Ching-Hua Wang**

1. INTRODUCTION.

Consider a system of quasilinear hyperbolic conservation laws

$$\frac{\partial U}{\partial t} + \frac{\partial f(U)}{\partial x} = 0 \tag{1.1}$$

where $U = U(x,t)$ is a n-vector, x the space variable and t the time variable. One of the interesting nonlinear features of the theory of conservation laws is that solutions attain very simple asymptotic states as t tends to infinity. The system is <u>strictly hyperbolic</u> if $\partial f(u)/\partial u$ has real and distinct eigenvalues $\lambda_1(u) < \lambda_2(u) < \ldots < \lambda_n(u)$ for each state u. For such a system, the solution of the initial value problem (1.1) and

$$U(x,0) = U_0(x) \tag{1.2}$$

tends to elementary waves as t tends to infinity [9], [10]. These waves are found by solving the Riemann problems (1.1) and

$$U(x,0) = \begin{cases} U_0(-\infty) & \text{for } x < 0 , \\ U_0(+\infty) & \text{for } x > 0 , \end{cases} \tag{1.3}$$

In particular, when the initial data (1.2) have a compact support then the solution tends to the zero state, [4], [2], [9]. This is so because waves combine and cancel as a consequence of the nonlinearity of the system and the entropy condition. The striking asymptotic behavior of the solution can be understood easily for scalar conservation law, [1], [8], [11].

When the system is <u>non-strictly hyperbolic</u>, that is, $\lambda_i(u)$ may equal $\lambda_j(u)$, $i \neq j$, for some states u, then waves pertaining to different characteristic families may not

*Department of Mathematics, University of Maryland, College Park, MD 20742

**Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, NY 10012

separate as time evolves. When this happens, nonlinear interactions of these waves may alter the asymptotic state. In this paper we study this problem for a system of two conservation laws whose Riemann problem has been studied, [5]. The system is derived from a model for elastic string. One of the characteristic speeds is linearly degenerate and the other genuinely nonlinear in the sense of [6]. We show that when the linear wave and the nonlinear wave in the solution of (1.1) and (1.3) are separated in the (x,t)-plane, then the asymptotic behavior of the solution of (1.1) and (1.2) is the same as that of a strictly hyperbolic system. In this case the asymptotic state consists of a traveling wave and a shock or rarefaction wave, c.f. [10]. On the other hand, when the linear wave in the solution of (1.1) and (1.3) is contained in the nonlinear wave then the corresponding traveling wave becomes substantially more regular than general traveling waves. This is so because of the strong interaction of the linear and nonlinear waves. It would be interesting to investigate the problem for more general systems where the interaction of nonlinear waves of different families occurs. For this further studies are necessary.

2. EQUATIONS AND RIEMANN PROBLEM.

The following two conservation laws are derived from a model of elastic strings, [5]:

$$\frac{\partial u}{\partial t} + \frac{\partial (\phi u)}{\partial x} = 0 \ ,$$

$$\frac{\partial v}{\partial t} + \frac{\partial (\phi v)}{\partial x} = 0 \ ,$$

(2.1)

where $\phi = \phi(u,v)$. Let (r,θ) be the polar coordinates,

$$r^2 = u^2 + v^2, \quad \tan \theta = v/u \ ,$$

and write $\phi = \phi(r,\theta)$. The characteristic speeds are

$$\lambda_1 = \phi$$

$$\lambda_2 = \phi + r \frac{\partial \phi}{\partial r} = \frac{\partial (r\phi)}{\partial r}$$

(2.2)

Thus (2.1) is not strictly hyperbolic on $\sum \equiv \{(r,\theta) : \frac{\partial \phi}{\partial r} = 0\} = \{(r,\theta) : \lambda_1 = \lambda_2\}$. The following assumptions on ϕ are consistent with physical considerations.

(A)$_1$ $\phi(r,\theta) \to \infty$ as $r \to 0$ or $r \to \infty$.

(A)$_2$ $\phi(r,\theta) > 0$,

(A)$_3$ $\dfrac{\partial(r\phi)}{\partial r} > 0$, $\dfrac{\partial^2(r\phi)}{\partial r^2} > 0$,

(A)$_4$ $\phi(\cdot,\theta)$ is convex for each fixed θ.

(A)$_1$ and (A)$_4$ imply that $\phi = C$, C any constant, is a simple closed curve. Also $\sum \equiv \{(r,\theta; = \frac{\partial\phi}{\partial r} = 0\}$ is a simple closed curve. (A)$_4$ implies that $\lambda_1 > \lambda_2$ inside $\sum$ and $\lambda_1 < \lambda_2$ outside $\sum$. The first characteristic speed λ_1 is always linearly degenerate in the sense of [6]. (A)$_3$ implies that λ_2 is genuinely nonlinear. The right eigenvectors r_i, $i = 1,2$, corresponding to λ_i are characterized by

$$\nabla\phi \cdot r_1 = 0$$
$$\nabla r \cdot r_2 = 0$$

Along the curve $\sum$, where $\lambda_1 = \lambda_2$, the system is diagonizable when

(A)$_5$ $\dfrac{\partial\phi}{\partial\theta} = 0$ on $\sum$.

In the elastic model it is reasonable to a assume that ϕ is a function of r only. In this case the above assumptions (A)$_1$ ~ (A)$_5$ are satisfied for general convex ϕ with appropriate growth rate at $r = 0$ and $r = \infty$.

To avoid the extreme case where two points on the string make contact, we will only deal with states in the following region:

$$\Lambda = \{U : r(U) > r_0, \ |\theta(u)| < \theta_0\}$$

for a fixed $r_0 > 0$ and $0 < \theta_0 < \frac{\pi}{2}$. The Riemann problem can be solved by similar methods as those for strictly hyperbolic systems. We have three kinds of elementary waves. A state U is connected to U_0 on the left by a 1-wave, which is always a contact discontinuity, if $U \in T(U_0)$:

$$T(U_0) \equiv \{U : \phi(U) = \phi(U_0)\} \ .$$

(U_0, U) forms a 2-shock wave (2-rarefaction wave) if $U \in S_2(U_0)$ ($U \in R_2(U_0)$):

$$S_2(U_0) \equiv \{U : \theta(U) = \theta(U_0), \ r(U) < r(U_0)\}$$
$$R_2(U_0) \equiv \{U : \theta(U) = \theta(U_0), \ r(U) > r(U_0)\}$$

The speed of a 2-shock wave (U_0, U_1) is

$$\sigma(U_0, U_1) = \frac{r_1 \phi_1 - r_2 \phi_2}{r_1 - r_2} .$$

To solve the Riemann problem one needs to know how the speed of a 2-wave is related to that of a 1-wave. Depending on the relative position of the waves in the solution of the Riemann problem $(U_\ell, U_\hbar)$ we have the following cases:

(I) U_ℓ and $U_\hbar$ are both outisde $\sum$ and $\phi(U_\hbar) > \phi(U_\ell)$.

The solution consists of a 1-wave (U_ℓ, U_m) followed by a 2-rarefaction wave $(U_m, U_\hbar)$. The state U_m is characterized by $\phi(U_m) = \phi(U_\ell)$ and $\theta(U_m) = \theta(U_\hbar)$ and is outside $\sum$.

(II) U_ℓ and $U_\hbar$ are both inside $\sum$ and $\phi(U_\ell) > \phi(U_\hbar)$.

The solution consists of a 2-rarefaction wave (U_ℓ, U_m) followed by a 1-wave $(U_m, U_\hbar)$. The state U_m is inside $\sum$ satisfying $\theta(U_m) = \theta(U_\ell)$ and $\phi(U_m) = \phi(U_\hbar)$.

(III) U_ℓ is inside $\sum$ and $U_\hbar$ is outside $\sum$.

The solution consists of a 2-rarefaction wave (U_ℓ, U_m) a 1-wave (U_m, U_n) and a 2-rarefaction wave (U_n, U_Ω). U_n and U_m are both on $\sum$ satisfying $\theta(U_m) = \theta(U_\ell)$ and $\theta(U_n) = \theta(U_\hbar)$.

The above three cases deal with solutions containing 2-rarefaction waves. The following two cases deal with solutions containing a 2-shock wave. Thus $U_\hbar$ is assumed to be "closer" to the origin than U_ℓ. Let U_m and U_n be defined by:

$$\phi(U_m) = \phi(U_\hbar), \ \theta(U_m) = \theta(U_\ell) ,$$
$$\phi(U_n) = \phi(U_\ell), \ \theta(U_n) = \theta(U_\hbar) .$$

The remaining two cases are:

(IV) $\phi(U_\ell) > \phi(U_\hbar)$

The solution consists of a 1-wave (U_ℓ, U_n) followed by a 2-shock wave $(U_n, U_\hbar)$. It is clear that $\sigma(U_n, U_m) > \phi(U_\ell)$ which is the speed of (U_ℓ, U_n).

(V) $\phi(U_\ell) < \phi(U_\hbar)$

The solution consists of a 2-shock wave (U_ℓ, U_m) followed by a 1-wave $(U_m, U_\hbar)$.

3. EXISTENCE OF THE SOLUTION.

The Glimm scheme, [3], for strictly hyperbolic systems can also be used to construct solutions for general hyperbolic systems. Choose a random sequence $\{a_k\}_{k=1}^{\infty}$, $-1 < a_k < 1$, and mesh length $\Delta x = h$, $\Delta t = s$, $h/s =$ constant satisfying the Courant-Friedrichs-Lewy condition:

$$\text{(C-F-L)} \quad \frac{h}{s} > \max_{U} |\lambda_i(U)|$$

for all U under consideration. The approximate solution $U_h(x,t) \equiv U_h(x,t; \{a_k\})$ is a step function of x for each fixed $t = ks$, $k = 0,1,2,\ldots$, with possible discontinuities at $x = jh$, $j + k =$ even. By resolving these discontinuities (see Section 2) $U_r(x,t)$ is defined for $t \in (ks,(k + 1)s)$. Elementary waves issued from $x = jr$, $j + k =$ even, do not interact before $t = (k + 1)s$ due to $(C - F - L)$ condition. At $t = (k + 1)s$, $U_r(x,t)$ is not a step function and the random sequence $\{a_k\}_{k=1}^{\infty}$ is used to approximate it by a step function:

$$U_h(x_1(k+1)s + 0) \equiv U_h((j + a_k)h_A(k+1)s - 0), \quad (j-1)h < x < (j+1)h, \quad j + k = \text{odd} .$$

This defines inductively the approximate solution $U_h(x,t)$ for all t.

The convergence of the approximate solution $\{U_h(x,t)\}$ as h tends to zero, is proved in two steps: First, one shows that $U_h(x,t)$ has bounded variation in x for each fixed t. This implies by diagonal process that $\{U_h(x,t)\}$ converges strongly in $L_1(x)$ for any rational t. To prove the convergence for all t, one needs the Lipschitz continuity of $L_1(x)$ in t. This follows from the estimate on the total variation in x of U_h and that $U_h(x,t)$ has a finite speed of propagation. For this the system has only to be hyperbolic, not necessarily strictly hyperbolic. Thus for the existence of the solution one need only to estimate the total variation in x of the approximate solution $U_h(x,t)$. It is convenient to introduce a new coordinate $(\bar{r},\theta)$ as follows: For any U in the region Λ let $\bar{U}$ be the unique state on the same side of $\sum$ with $\phi(U) = \phi(\bar{U})$ and $\theta(\bar{U}) = 0$. We set

$$\bar{r}(U) = r(\bar{U}) .$$

It follows from the assumptions $(A)_1 \sim (A)_5$ of Section 2 that the transformation $(\bar{r},\theta) \to (u,v)$ is nonsingular for $U = (u,v)$ in a bounded region of Λ. Note that a 1-wave takes values along $\bar{r} = $ constant and a 2-wave takes values along $\theta = $ constant. The strength of a 1-wave is defined by the jump of θ across the wave; and the strength of a 2-wave by the jump of $\bar{r}$ across it. The following lemmas follow directly from the recipe for solving the Riemann problem presented in the last section. We omit details.

Lemma 1: Suppose that the initial data $U(x,0)$ stay in a bounded region

$$\Omega \equiv \left\{ U : 0 < r_1 < \bar{r}(U) < r_2 < \infty, \ |\theta|(U) < \theta_0 < \frac{\pi}{2} \right\}.$$

Then any approximate solution $U_h(x,t)$ also stay in Ω for any (x,t).

Lemma 2: Suppose that $U_\ell, U_m, U_\hbar$ are three states in Ω, then the total strength of waves in the Riemann problem $(U_\ell, U_\hbar)$ is no larger than the sum of the total strength of waves in the Riemann problem (U_ℓ, U_m) and the total strength of waves in the Riemann problem $(U_m, U_\hbar)$.

Lemma 1 shows that there are bounded invariant regions for the solution and so in particular the $(C - F - L)$ can easily be satisfied. Lemma 2 shows that nonlinear interactions do not cause an increase in the strength of waves. This is sufficient to obtain the desired estimate on the variation in x of the approximate solution $U_\hbar(x,t)$, (c.f. [13]). Thus we have the following existence theorem.

Theorem: Suppose that assumptions $(A)_1 \sim (A)_5$ hold and the initial data $U(x,0)$ stay in a bounded region $\Omega \equiv \left\{ U : 0 < r_1 < \bar{r}(U) < r_2 < \infty, \ |\theta(U)| < \theta_0 < \frac{\pi}{2} \right\}$ and have bounded variation locally in x. Then the initial value problem for (2.1) has a global solution $U(x,t)$ which stays in Ω and has bounded variation locally in x.

4. ASYMPTOTIC BEHAVIOR OF THE SOLUTION.

In this section we assume that $\varphi = \varphi(r)$. The advantage of this assumption is the weaker coupling of the system (2.1) and so the behavior of 2-waves can be studied independently. Across a 1-wave, the value of r is unchanged. A 2-rarefaction wave has speed $(r\phi)_r$ and a 2-shock wave (U_0, U_1) has speed

$$\sigma(U_0, U_1) = \frac{r_1 \phi_1 - r_0 \phi_0}{r_1 - r_0}$$

Moreover, across a 2-wave θ is constant. Therefore the behavior of r, and hence the behavior of 2-waves, can be described by the scalar conservation law

$$\frac{\partial r}{\partial t} + \frac{\partial (r\phi(r))}{\partial x} = 0 . \tag{4.1}$$

A 1-wave propagates with speed ϕ and so 1-waves are described by

$$\frac{\partial \theta}{\partial t} + \phi(r) \frac{\partial \theta}{\partial x} = 0 . \tag{4.2}$$

The identities (4.1) and (4.2) can be derived from the system (2.1) directly. For instance, multiplying the first equation of (2.1) by u and the second equation by v and summing them up, one obtains (4.1). This procedure is justified for smooth solutions (u,v). However, in general when an equation is derived from a system of conservation laws through nonlinear transformations, the derived equation may not be a consequence of these conservation laws in the weak sense. In the present situation the procedure is justified because the jump condition and entropy condition for (4.1) and (4.2) are consistent with those for (2.1). Our strategy is to study the behavior of 2-waves using (4.1) and, having obtained the behavior of r, use (4.2) to study the behavior of 1-waves.

From now on we assume that the initial data satisfy

$$U(x,0) = \begin{cases} U_\ell & \text{for } x < -s/2 \\ U_h & \text{for } x > s/2 . \end{cases}$$

for some constant states U_ℓ and U_h. We denote by $\bar{U}(x,t)$ the solution of the corresponding Riemann problem (2.1) with

$$U(x,0) = \begin{cases} U_\ell & \text{for } x < 0 , \\ U_h & \text{for } x > 0 . \end{cases} \tag{4.3}$$

The behavior of 2-waves is described by (4.1) with

$$r(x,0) = \begin{cases} r_\ell & \text{for } x < -s/2 \\ r_h & \text{for } x > s/2 . \end{cases} \tag{4.3}'$$

There two cases; [8]:

Case 1: $r_\ell > r_{\hbar}$

There exists $T > 0$ such that for $t \geqslant T$, the solution $r(x,t)$ of (4.1) and (4.3)'
is a shock wave $(r_\ell, r_{\hbar})$:

$$r(x,t) = \begin{cases} r_\ell & \text{for} \ \ x - x_0 < \sigma t \\ r_{\hbar} & \text{for} \ \ x - x_0 > \sigma t \end{cases} , \qquad t > T \ .$$

$$\sigma \equiv \frac{r_{\hbar}\phi(r_{\hbar}) - r_\ell\phi(r_\ell)}{r_{\hbar} - r_\ell}$$

$$x_0 \equiv \frac{1}{r_{\hbar} - r_\ell} \int_{-s/2}^{s/2} \left[\frac{r_{\hbar} + r_\ell}{2} - r(x,0)\right]dx \ .$$

Case 2: $r_\ell < r_{\hbar}$

The solution $r(x,t)$ tends to a generalized N-wave defined as follows:

$$N(x,t) \equiv N(x,t;p,q;r_\ell,r_\ell)$$

$$= \begin{cases} r_\ell & \text{for} \ \ x \leqslant \lambda_2(r_\ell)t - \sqrt{-2p\lambda_2'(r_\ell)t} \equiv \bar{x}(t) \\ r_{\hbar} & \text{for} \ \ x \geqslant \lambda_2(r_\ell)t + \sqrt{2q\lambda_2'(r_{\hbar})t} \equiv \bar{x}_{\hbar}(t) \\ r(\xi) & \text{otherwise;} \ \ \xi = x/t \end{cases}$$

$$p \equiv \min_x \int_{-\infty}^x [r(y,0) - r_\ell]dy, \quad q \equiv \max_x \int_x^\infty [r(y,0) - r_{\hbar}]dy$$

More precisely, there exist Lipschitz continuous curves $x = x_\ell(t)$ and $x_{\hbar}(t)$ through
$(-s/2,0)$ and $(s/2,0)$, respectively, with the following properties:

(i) $|x_{\hbar}(t) - \bar{x}_{\hbar}(t)| + |x_\ell(t) - \bar{x}_\ell(t)| = O(S)$.

(ii) $|r(x,t) - N(x,t)| = O(1)t^{-1}$ for (x,t) between $x_\ell(t)$ and $x_{\hbar}(t)$ and also
between $\bar{x}_\ell(t)$ and $\bar{x}_{\hbar}(t)$.

(iii) $|r(x,t) - N(x,t)| = O(1)t^{-\frac{1}{2}}$ either (x,t) lies between $x_\ell(t)$ and $\bar{x}_\ell(t)$
or between $x_{\hbar}(t)$ and $\bar{x}_{\hbar}(t)$.

(iv) $r(x,t) = \begin{array}{l} r_\ell \ \ \text{for} \ \ x < x_\ell(t) \ , \\ r_{\hbar} \ \ \text{for} \ \ x > x_{\hbar}(t) \ . \end{array}$

We now use the above known results to study the behavior of 1-waves. When waves of different families in the solution of the Riemann problem (2.1) and (4.3) are separated, the asymptotic behavior of the solution $U(x,t)$ of (2.1) and (2.2) is similar to that of solutions of a strictly hyperbolic system. In this case 1-waves tend to a traveling wave. We illustrate this by investigating Case (V) of Section 2. Thus we have $\phi(r_\ell) < \phi(r_\hbar)$. From (4.2) characteristic curves for 1-waves are given by:

$$\frac{dx}{dt} = \phi(r(x,t)) \ .$$

Through $(-s/2,0)$ and $(s/2,0)$, respectively, we draw two characteristics X_1 and X_2. We know from the above discussion of 2-waves, (case 1), that $r(x,t)$ is a shock wave for $t > T$. The speed of the shock wave is σ which is less than $\phi(r_\ell)$ because $\phi(r_\ell) < \phi(r_\hbar)$. Consequently the 1-characteristic X_1 lies to the right of the shock wave after time T_1, T_1 finite. After time T_1 there exists no 1-wave either to the left of X_1 or to the right of X_2, and, between X_1 and X_2, θ is constant along $dx/dt = \phi(r_\hbar)$ according to (4.2). Thus after time T_1, 1-waves become a traveling wave between X_1 and X_2 and with speed $\phi(r_\hbar)$. The value of θ to the left of X_1 is θ_ℓ and to the right of X_2 is $\theta_\hbar$.

The situation is more complicated when waves of different families do not separate. We exemplify this by investigating Case III in Section 2. In this case, the solution of the Riemann problem $(U_\ell, U_\hbar)$ is a contact discontinuity (U_m, U_n) sandwiched by two 2-rarefaction waves (U_ℓ, U_m) and $(U_n, U_\hbar)$. The states U_m and U_n are on $\sum$ and so $\lambda_1(U_m) = \lambda_2(U_m) = \lambda_1(U_n) = \lambda_2(U_m)$. We consider the case when $U_\ell \neq U_m$ and $U_n \neq U_\hbar$. Note that λ_1 attains an absolute minimum on $\sum$ and so the characteristic curve for (4.2) always has speed larger than or equal to $\phi(r_m) = \phi(r_n)$. Suppose that

$$e(x) \equiv \int_0^x [r_0(y) - r_m]dy$$ attains a minimum at $x = x_0$. Then the characteristic line through $(x_0, 0)$ for (4.1) exists for all $t > 0$, [7]. In this case the characteristic curve through $(x_0, 0)$ for (4.2) coincides with that for (4.1). In fact only such characteristic curves for (4.2) may travel with the minimal speed $\phi(r_m)$ for all $t > 0$. Thus when $I(x)$ takes minima at $x = \alpha'$ and $x = \alpha''$ and not for any $x \in (\alpha', \alpha'')$, then

all characteristic curves through $(x,0)$, $\alpha' < x < \alpha''$, tend to the characteristic line $x = x_0 + \phi(r_m)t$ and so

$$\lim_{t \to \infty} \theta(x + \phi(r_m)t, t) = \theta(\alpha' + 0, 0) \quad \text{for} \quad x \in (\alpha', \alpha'') . \tag{4.4}$$

It is clear that

$$m(r_m) \equiv \{x_0 : \int_0^x [r_0(x) - r_m]dx \quad \text{attains minimum at} \quad x = x_0\}$$

is a closed set. Denote by x_m and x_M, respectively, the smallest and largest numbers in $m(r_m)$. The set $\ell(r_m) \equiv [x_m, x_M] - m(r_m)$ is an open set. We have just described the asymptotic behavior of $\theta(x,t)$ for $x = x_0 + \phi(r_m)t$, x_0 in a component of $\ell(r_m)$. When $m(r_m)$ contains an interval (a,b) we have

$$r(x,t) = r_\ell \quad \text{for} \quad \phi(r_m)t + a < x < \phi(r_m)t + b ,$$
$$\theta(x,t) = \theta(x - \phi(r_m)t, 0), \quad \text{for} \quad \phi(r_m)t + a < x < \phi(r_m)t + b . \tag{4.5}$$

For $x < x_m + \phi(r_m)t$, all characteristic curves for (4.2) tend to the characteristic line through $(x_m, 0)$. Since $\theta(x,0) = \theta_\ell$ for $x < -s/2$ we have

$$\lim_{t \to \infty} \theta(x + \phi(r_m)t, t) = \theta_\ell \quad \text{for} \quad x < x_m . \tag{4.6}$$

We have seen that the initial values of θ restricted to $(-\infty, x_M)$ may not be carried to $t = \infty$ and the asymptotic shape of $\theta(x,t)$, $x - \phi(r_m)t \in (-\infty, r_M)$, is in general a step function. On the other hand, the initial data $\theta(x,0)$, $x > M$, are in general carried to $t = \infty$ and the asymptotic behavior of $\theta(x,t)$, $x - \phi(r_m)t \in (x_M, \infty)$, is a traveling wave taking values $\{\theta(x,0), x > x_M\}$. We now show that this traveling wave has a finite width. For this, we draw a 1-characteristic X through $(s/2, 0)$ and estimate the asymptotic location and speed of X. Since $r(x,t)$ tends to an N-wave and all 2-shock waves, except the one issued from $s/2$, decay at the rate $1/t$, we will carry out the analysis by supposing that $r(x,t)$ is a centered rarefaction wave. This is done for simplicity; the general case can be dealt with similarly. Thus it follows from the structure of centered rarefaction wave that when $r = r_0$ at $\{t = t_0\} \cap X$ and $r = r_0 - \Delta r$ at $\{t = t_0 + \Delta t\} \cap X$ then

$$[\lambda_2(r) - \lambda_2(r - \Delta r)]t \stackrel{\sim}{=} [\lambda_2(r) - \lambda_1(r)]\Delta t$$

This yields an ordinary differential equation for the value of r along X:

$$\frac{d\lambda_2(r(t))}{dt} = -[\lambda_2(r(t)) - \lambda_1(r(t))]/t \qquad (4.7)$$

Note that $\lambda_2(r) > \lambda_1(r)$ for r in the region under consideration, i.e. $r > r_m$. Thus the above identity implies that

$$r(t) \to r_m \quad \text{as} \quad t \to \infty .$$

The rate of this convergence is determined as follows: Consider

$$\frac{d\xi(t)}{dt} = -\xi(t)/t$$

$$\xi(t) \equiv \lambda_2(r(t)) - \lambda_1(r(t)) . \qquad (4.8)$$

It is clear that

$$\xi(t) = O(1) \frac{1}{t} \quad \text{as} \quad t \to \infty .$$

Note that as $r(t)$ tends to r_m

$$|\lambda_2(r(t)) - \lambda_2(r_m)| \sim |r(t) - r_m| ,$$

$$|\lambda_1(r(t)) - \lambda_1(r_m)| \sim |r(t) - r_m|^2 ,$$

and so $r(t)$ and $\xi(t)$ have the same qualitative behavior. We thus have:

$$|r(t) - r_m| = O(1) \frac{1}{t} ,$$

$$|\lambda_1(t) - \lambda_1(r_m)| = O(1) \frac{1}{t^2} . \qquad (4.9)$$

This implies that X tends to a straight line $x - \phi(r_m)t = \text{constant}$ at the rate $1/t$. In particular $\theta(x,t)$, $x > x_M + \phi(r_M)t$, tends to a traveling wave of finite width. Note that the speed of X is always larger than $\phi(r_m)$ and so the distance between

$\{(x,t) : x = x_M + \phi(r_m)t\}$ and X is an increasing function of time. Consequently, except for the exceptional case where $X_M = S/2$, the asymptotic distance of these two curves is finite and positive. This completes the description of the asymptotic state of $\theta(x,t)$.

In the above arguments we have assumed that $\theta(x,t)$ is constant along the
2-characteristics $dx/dt = \phi(r(x,t))$ and that 2-characteristics are defined for all
$t > 0$. This is so because across a 2-shock wave θ is unchanged and the 1-characteristic
speeds ϕ on both sides of the shock are either greater or less than the shock speed.

The above arguments can be applied to treat other cases in Section 2. We briefly
state the asymptotic results in the following theorem.

<u>Theorem 4.1</u>: Suppose that φ is a function of r and hypothesis $(A)_i$, $i = 1,2,...,5$,
hold and the initial data $U(x,0)$ equals U_ℓ for $x < -S/2$ and $U_{\hbar}$ for $x > S/2$. Then
the asymptotic behavior of the solution of (2.1) is as follows:

(i) The behavior of 2-waves is described by (4.1). Thus 2-waves tend to a single 2-
shock wave when $r_\ell > r_{\hbar}$ and to a N-wave when $r_\ell < r_{\hbar}$.

(ii) When 1-waves and 2-waves in the solution of the Riemann problem $(U_\ell, U_{\hbar})$ for
(2.1) are separated, 1-waves tend to a traveling wave of finite width in finite time. The
traveling wave assumes all the values of $\theta(x,0)$. The same also holds when the 2-wave in
the solution of (U_ℓ, U_Ω) is a shock wave.

(iii) When the 2-wave in the solution of $(U_\ell, U_{\hbar})$ is a rarefaction wave which
contacts the 1-wave in $(U_\ell, U_{\hbar})$, 1-waves tend to a traveling wave as t tends to
infinity. The traveling wave is a combination of a step function and a general traveling
wave. Moreover, the traveling wave does not assume all the initial values $\theta(x,0)$ and is
described by (4.4), (4.5) and (4.6).

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We study a system of quasilinear hyperbolic conservation laws which is hyperbolic but not strictly hyperbolic. Such systems arise naturally in continuum mechanics such as elastic, multiphase flows. We are interested mainly in the large time behavior of the solution. Due to the nonlinearity of the system and the entropy condition, solutions converge to very simple
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ABSTRACT (continued)

elementary waves. Nonstrict hyperbolicity of the system may cause a stronger
nonlinear interactions between waves pertaining to different families; in
particular, such interactions may regularize linear waves in the solution.
The solutions are constructed using the random choice method.